

# All Universal Coverings Are Spanier Spaces

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## Abstract

In this paper, we study necessary and sufficient conditions for existence of categorical universal coverings using open covers of a given space  $X$ . In fact, we define several homotopy theoretic conditions which we then prove are equivalent to the existence of a categorical universal covering space. As an application, we show that all universal coverings of a connected and locally path connected space are Spanier spaces.

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## 1. Introduction and motivation

A continuous map  $p : \tilde{X} \rightarrow X$  is a *covering* of  $X$ , and  $\tilde{X}$  is called a *covering space* of  $X$ , if for every  $x \in X$  there exists an open subset  $U$  of  $X$  with  $x \in U$  such that  $U$  is *evenly covered* by  $p$ , that is,  $p^{-1}(U)$  is a disjoint union of open subsets of  $\tilde{X}$  each of which is mapped homeomorphically onto  $U$  by  $p$ . In the classical covering theory, if  $X$  is connected and locally path connected, then one will be able to classify all path connected covering spaces of  $X$  and find among them a universal covering space, that is, a covering  $p : \tilde{X} \rightarrow X$  with the property that for every covering  $q : \tilde{Y} \rightarrow X$  by a path connected space  $\tilde{Y}$  there is a unique covering  $r : \tilde{X} \rightarrow \tilde{Y}$  such that  $r \circ q = p$ .

If  $X$  is a connected and locally path-connected space, we have the following well-known result (see [6]).

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*Every simply connected covering space of  $X$  is a universal covering space. Moreover,  $X$  admits a simply connected covering space if and only if  $X$  is semi-locally simply connected.*

E.H. Spanier [6] classified path connected covering spaces of a space  $X$  using some subgroups of the fundamental group of  $X$ , recently named Spanier groups (see [2]). If  $\mathcal{U}$  is an open cover of  $X$ , then the subgroup of  $\pi_1(X, x)$  consisting of all homotopy classes of loops that can be represented by a product of the following type

$$\prod_{j=1}^n \alpha_j * \beta_j * \alpha_j^{-1},$$

where the  $\alpha_j$ 's are arbitrary paths starting at the base point  $x$  and each  $\beta_j$  is a loop inside one of the neighborhoods  $U_j \in \mathcal{U}$ , is called the *Spanier group with respect to  $\mathcal{U}$* , and denoted by  $\pi(\mathcal{U}, x)$  [2, 6]. The following theorem is an interesting result on the above notion.

**Theorem 1.1.** ([6]). *For a connected, locally path connected space  $X$ , if there exists an open cover  $\mathcal{U}$  of  $X$  such that  $\pi(\mathcal{U}, x) \leq H$ , where  $H$  is a subgroup of  $\pi_1(X, x)$ , for  $x \in X$ , then there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$ .*

Classification of covering spaces for the spaces that are not locally nice is not as pleasant. H. Fischer and A. Zastrow in [3] defined a generalized regular covering which enjoys most of the usual properties of classical coverings with the possible exception of evenly covered neighborhoodness. Also, the authors [4, 5, 7] classified covering spaces of some “wild” spaces, for example non-homotopically Hausdorff spaces and non-homotopically path Hausdorff spaces.

The chromatic role of the open covers in the classification of covering spaces is also seen in [5], where the authors use the Spanier group of a space  $X$  (see [2, 6])

$$\pi_1^{sp}(X, x) = \bigcap_{\text{open covers } \mathcal{U}} \pi(\mathcal{U}, x),$$

in order to introduce a new type of coverings of  $X$  that are universal. The authors [5] introduce a Spanier covering  $p : \tilde{X} \rightarrow X$  which has the property  $p_*\pi_1(\tilde{X}, \tilde{x}) = \pi_1^{sp}(X, x)$  and show that these coverings exist if and only if  $X$  is a semi-locally Spanier space, i.e, every point  $x \in X$  has a neighborhood  $U$  such that  $i_*\pi_1(U, x) \leq \pi_1^{sp}(X, x)$ . In this case,  $\tilde{X}$  is a Spanier space which means that  $\pi_1(\tilde{X}, \tilde{x}) = \pi_1^{sp}(\tilde{X}, \tilde{x})$  (see [5]).

In this paper, we introduce coverable spaces that are spaces in which for every subgroup  $H$  of their fundamental groups containing the Spanier group, there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$ . Then, using the properties of

open covers, we show that all universal coverings of a connected and locally path connected space are Spanier coverings and in this case, the base space is semi-locally Spanier space. Also, we show that a space is coverable if and only if the Spanier groups of open covers are stable for arbitrary small open covers.

Throughout this article, all homotopies between two paths are relative to end points,  $X$  is a connected and locally path connected space with the base point  $x \in X$ , and  $p : \tilde{X} \rightarrow X$  is a path connected covering of  $X$  with  $\tilde{x} \in p^{-1}(\{x\})$  as the base point of  $\tilde{X}$ . Also, by universal covering, we mean the categorical universal covering.

## 2. Main results

Spanier [6] used a family of subgroups of fundamental groups related to open covers of a given space  $X$  for classification of covering spaces of  $X$ . He proved that for a subgroup  $H \leq \pi_1(X, x)$ , when  $X$  is connected and locally path connected, there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$  if there exists an open cover  $\mathcal{U}$  such that  $\pi(\mathcal{U}, x) \leq H$ . First, we prove the converse of the Spanier result.

**Proposition 2.1.** *Suppose  $X$  is a connected, locally path connected space, and there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$ . Then there exists an open cover  $\mathcal{U}$  of  $X$  such that  $\pi(\mathcal{U}, x) \leq H$ .*

*Proof.* Let  $\mathcal{U}$  be an open cover of  $X$  containing evenly covered open neighborhoods. Assume  $[\alpha * \beta * \alpha^{-1}] \in \pi(\mathcal{U}, x)$ , where  $\beta$  is a loop in  $U \in \mathcal{U}$  with  $y := \alpha(1)$  as base point and  $\alpha$  is a path from  $x$  to  $y$ . Let  $\tilde{\alpha}$  be the lifting of  $\alpha$  with base point  $\tilde{x}$  and  $V$  be the homeomorphic copy of  $U$  in  $\tilde{X}$  that contains  $\tilde{y} := \tilde{\alpha}(1)$ . Then  $\tilde{\beta} := p|_V^{-1} \circ \beta$  is a loop in  $V$  with base point  $\tilde{y}$ . Hence  $[\tilde{\alpha} * \tilde{\beta} * \tilde{\alpha}^{-1}] \in \pi_1(\tilde{X}, \tilde{x})$  which implies that  $[\alpha * \beta * \alpha^{-1}] \in p_*\pi_1(\tilde{X}, \tilde{x})$ .  $\square$

**Corollary 2.2.** *Let  $X$  be a connected, locally path connected space and  $H \leq \pi_1(X, x)$ , for  $x \in X$ . Then there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$  if and only if there exists an open cover  $\mathcal{U}$  of  $X$  in which  $\pi(\mathcal{U}, x) \leq H$ .*

**Notation 2.3.** *For a space  $X$  and  $H \leq \pi_1(X, x)$ , by  $\tilde{X}_H$  we mean a covering space of  $X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$ , where  $\tilde{x} \in p^{-1}(x)$  and  $p : \tilde{X} \rightarrow X$  is the corresponding covering map.*

For two open covers  $\mathcal{U}, \mathcal{V}$  of  $X$ , we say that  $\mathcal{V}$  refines  $\mathcal{U}$  if for every  $V \in \mathcal{V}$ , there exists  $U \in \mathcal{U}$  such that  $V \subseteq U$ . Using the properties of open covers and the definition of the Spanier groups with respect to open covers, we have the following facts which have been also remarked in [6].

**Proposition 2.4.** *Let  $\mathcal{U}, \mathcal{V}$  be open covers of a space  $X$ . Then the following statements hold.*

- (i) *If  $\mathcal{V}$  refines  $\mathcal{U}$ , then  $\pi(\mathcal{V}, x) \subseteq \pi(\mathcal{U}, x)$ , for every  $x \in X$ .*
- (ii)  *$\pi(\mathcal{U}, x)$  is a normal subgroup of  $\pi_1(X, x)$ .*
- (iii) *If  $\alpha$  is a path in  $X$ , then  $\varphi_{[\alpha]}(\pi(\mathcal{U}, \alpha(0))) = \pi(\mathcal{U}, \alpha(1))$ , where  $\varphi_{[\alpha]}([\beta]) = [\alpha^{-1} * \beta * \alpha]$ .*

**Theorem 2.5.** *For a connected and locally path connected space  $X$ , let  $H, K \leq \pi_1(X, x)$ . Then  $\tilde{X}_H$  and  $\tilde{X}_K$  exist if and only if  $\tilde{X}_{H \cap K}$  exists.*

*Proof.* By Corollary 2.2, existence of  $\tilde{X}_H$  and  $\tilde{X}_K$  implies the existence of open covers  $\mathcal{U}$  and  $\mathcal{V}$  of  $X$  such that  $\pi(\mathcal{U}, x) \leq H$  and  $\pi(\mathcal{V}, x) \leq K$ . Let  $\mathcal{U} \cap \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$  which is a refinement of  $\mathcal{U}$  and  $\mathcal{V}$ . Hence  $\pi(\mathcal{U} \cap \mathcal{V}) \subseteq \pi(\mathcal{U}) \subseteq H$  and  $\pi(\mathcal{U} \cap \mathcal{V}) \subseteq \pi(\mathcal{V}) \subseteq K$  which implies that  $\pi(\mathcal{U} \cap \mathcal{V}) \subseteq H \cap K$ . Therefore, there exists  $\tilde{X}_{H \cap K}$ . The converse is trivial.  $\square$

The above theorem shows that intersections of open covers of a space  $X$  are important in the existence of new coverings of  $X$ . So it is interesting to find the role of the intersection of all open covers. For this, we use the Spanier groups.

**Proposition 2.6.** ([5]). *If  $p : \tilde{X} \rightarrow X$  is a covering of  $X$ , then  $\pi_1^{sp}(X, x) \leq p_*\pi_1(\tilde{X}, \tilde{x})$ , for every  $x \in X$ .*

A desirable fact in the category of coverings of a space  $X$  is the existence of  $\tilde{X}_H$ , for every subgroup  $H \leq \pi_1(X, x)$ . We characterize spaces with this property as follows.

**Definition 2.7.** *We call a topological space  $X$  a coverable space if  $\tilde{X}_H$  exists, for every subgroup  $\pi_1^{sp}(X, x) \leq H \leq \pi_1(X, x)$ .*

Note that the above notion does not depend on the point  $x$ . Also, since the image subgroups of all the coverings contain  $\pi_1^{sp}(X, x)$ , eliminating the condition  $\pi_1^{sp}(X, x) \leq H$  from the above definition is meaningless.

The following result is well-known in the classical covering theory.

**Corollary 2.8.** *Every connected, locally path connected and semi-locally simply connected space is coverable.*

**Proposition 2.9.** *Let  $X$  be a connected, locally path connected and coverable space. Then  $\pi_1^{sp}(X, x) = 1$  if and only if  $X$  is semi-locally simply connected.*

*Proof.* Since  $X$  is coverable, there exists a covering  $p : \tilde{X} \rightarrow X$  such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = \pi_1^{sp}(X, x) = 1$  and hence  $\tilde{X}$  is simply connected which implies that  $X$  is semi-locally simply connected. The converse holds by Proposition 2.6.  $\square$

The Hawaiian earing space,  $HE$ , is a famous example of a space which is not semi-locally simply connected. Also, Spanier group of the Hawaiian earing space is trivial since if  $\mathcal{U}_n$ 's are open covers of the Hawaiian earing by open disk with diameter  $1/n$ , for every  $n \in \mathbb{N}$ , then  $\pi_1^{sp}(HE, 0) \leq \bigcap_{n \in \mathbb{N}} \pi(\mathcal{U}_n, 0) = 1$ . Hence we have the following corollary.

**Corollary 2.10.** *The Hawaiian earing space is not coverable.*

**Theorem 2.11.** *A space  $X$  is coverable if and only if  $\tilde{X}_{\pi_1^{sp}(X, x)}$  exists.*

*Proof.* The necessity comes from the definition. For the sufficiency, let  $\pi_1^{sp}(X, x) \leq H \leq \pi_1(X, x)$ . By Proposition 2.1, since  $\tilde{X}_{\pi_1^{sp}(X, x)}$  exists, there is an open cover  $\mathcal{U}$  of  $X$  such that  $\pi(\mathcal{U}, x) \leq \pi_1^{sp}(X, x)$ . Hence, using Corollary 2.2,  $\tilde{X}_H$  exists.  $\square$

By the proof of Proposition 2.1, if  $p : \tilde{X} \rightarrow X$  is a covering such that  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$  and  $\mathcal{U}$  is the open cover of  $X$  by evenly covered open neighborhoods, then  $\pi(\mathcal{U}) \leq H$ . Moreover, if  $H = \pi_1^{sp}(X, x)$ , then we have the following interesting result.

**Proposition 2.12.** *If  $\tilde{X}_{\pi_1^{sp}(X, x)}$  exists, then there exists an open covering  $\mathcal{U}$  of  $X$  such that for every refinement  $\mathcal{V} \subseteq \mathcal{U}$  we have  $\pi(\mathcal{U}) = \pi(\mathcal{V})$ .*

*Proof.* Assume  $p : \tilde{X}_{\pi_1^{sp}(X, x)} \rightarrow X$  is a covering and  $\mathcal{U}$  is an open cover of  $X$  by evenly covered open neighborhoods. Since  $p_*\pi_1(\tilde{X}_{\pi_1^{sp}(X, x)}, \tilde{x}) = \pi_1^{sp}(X, x)$ ,  $\pi(\mathcal{U}, x) \subseteq \pi_1^{sp}(X, x)$  and hence  $\pi(\mathcal{U}, x) = \pi_1^{sp}(X, x)$ . Also, for every refinement  $\mathcal{V}$  of  $\mathcal{U}$ , we have  $\pi(\mathcal{V}, x) \subseteq \pi(\mathcal{U}, x)$  which implies that  $\pi(\mathcal{V}, x) = \pi_1^{sp}(X, x)$ . Hence the result holds.  $\square$

**Definition 2.13.** *We say that an open cover  $\mathcal{U}$  of a space  $X$  is  $\pi$ -stable if  $\pi(\mathcal{U}) = \pi(\mathcal{V})$ , for every refinement  $\mathcal{V}$  of  $\mathcal{U}$ .*

Note that if  $\mathcal{U}$  and  $\mathcal{V}$  are two  $\pi$ -stable open covers of  $X$ , then  $\pi(\mathcal{U}) = \pi(\mathcal{V})$ .

**Theorem 2.14.** *Let  $\mathcal{U}$  be an open cover of a space  $X$ . Then  $\mathcal{U}$  is  $\pi$ -stable if and only if  $\pi_1^{sp}(X, x) = \pi(\mathcal{U}, x)$ .*

*Proof.* Let  $\mathcal{U}$  be a  $\pi$ -stable open cover of  $X$ . By the definition  $\pi_1^{sp}(X, x) \leq \pi(\mathcal{U}, x)$ . For the reverse containment, let  $\mathcal{V}$  be an arbitrary open cover of  $X$ . Then  $\mathcal{U} \cap \mathcal{V}$  is a refinement of  $\mathcal{U}$  and hence  $\pi(\mathcal{U}) = \pi(\mathcal{U} \cap \mathcal{V}) \leq \pi(\mathcal{V})$ . Therefore  $\pi(\mathcal{U}) \leq \pi_1^{sp}(X, x)$ , as desired. Conversely, let  $\mathcal{U}$  be an open cover of  $X$  such that  $\pi_1^{sp}(X, x) = \pi(\mathcal{U})$ , and  $\mathcal{V}$  be a refinement of  $\mathcal{U}$ . Then  $\pi(\mathcal{V}) \leq \pi(\mathcal{U}) = \pi_1^{sp}(X, x)$  which implies that  $\pi(\mathcal{V}) = \pi_1^{sp}(X, x) = \pi(\mathcal{U})$ .  $\square$

**Theorem 2.15.** *A space  $X$  is coverable if and only if there exists an open cover  $\mathcal{U}$  of  $X$  such that  $\pi_1^{sp}(X, x) = \pi(\mathcal{U}, x)$ .*

*Proof.* Using Corollary 2.2, existence of an open cover  $\mathcal{U}$  such that  $\pi(\mathcal{U}) = \pi_1^{sp}(X, x)$  implies the existence of  $\tilde{X}_{\pi_1^{sp}(X, x)}$ . Hence by Theorem 2.11,  $X$  is coverable. Conversely, assume that  $X$  is coverable. The result comes from the definition, Proposition 2.12 and Theorem 2.14.  $\square$

The following theorem shows the importance of a universal covering for the existence of other coverings and vice versa.

**Theorem 2.16.** *A space  $X$  has a universal covering if and only if  $X$  is coverable.*

*Proof.* If  $X$  is coverable, then by the definition,  $\tilde{X}_{\pi_1^{sp}(X, x)}$  exists. Since by Proposition 2.6,  $\tilde{X}_{\pi_1^{sp}(X, x)}$  is a universal covering space, the result holds. Conversely, assume that  $p : \tilde{X} \rightarrow X$  is a universal covering of  $X$  and  $p_*\pi_1(\tilde{X}, \tilde{x}) = H$ . We claim that for every open cover  $\mathcal{U}$  of  $X$ ,  $H \leq \pi(\mathcal{U}, x)$ . For, if  $q : \tilde{X}_{\pi(\mathcal{U})} \rightarrow X$  is the covering such that  $q_*\pi_1(\tilde{X}_{\pi(\mathcal{U})}) = \pi(\mathcal{U})$ , then by the universal property of  $p : \tilde{X} \rightarrow X$

$$H = p_*\pi_1(\tilde{X}, \tilde{x}) \leq q_*\pi_1(\tilde{X}_{\mathcal{U}}) = \pi(\mathcal{U}).$$

Hence  $H \leq \pi_1^{sp}(X, x)$  which implies that  $H = \pi_1^{sp}(X, x)$ . Thus the covering space  $\tilde{X}_{\pi_1^{sp}(X, x)}$  exists and hence by Theorem 2.11,  $X$  is coverable.  $\square$

**Definition 2.17.** *A space  $X$  is called a semi-locally Spanier space if for every point  $x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $i_*\pi_1(U, x) \leq \pi_1^{sp}(X, x)$ .*

**Lemma 2.18.** *A space  $X$  is semi-locally Spanier space if and only if there exist an open cover  $\mathcal{U}$  of  $X$  such that  $\pi(\mathcal{U}, x) = \pi_1^{sp}(X, x)$ , for every  $x \in X$ .*

*Proof.* Using Proposition 2.4 (iii) and the definition of  $\pi(\mathcal{U}, x)$ , the result holds.  $\square$

**Theorem 2.19.** *For a connected and locally path connected space  $X$ , the following statements are equivalent.*

- (i)  $X$  is coverable.
- (ii)  $X$  has a universal covering space.
- (iii)  $X$  has a  $\pi$ -stable open cover.
- (iv)  $X$  is a semi-locally Spanier space.
- (v)  $\pi_1^{sp}(X, x)$  is an open subgroup of  $\pi_1^{top}(X, x)$ .

*Proof.* (i)  $\Leftrightarrow$  (ii) It follows from Theorem 2.16.

(i)  $\Leftrightarrow$  (iii) It follows from Theorems 2.14 and 2.15.

(i)  $\Leftrightarrow$  (iv) It follows from Theorem 2.15 and Lemma 2.18.

(i)  $\Leftrightarrow$  (v) By [1, Theorem 5.5], the connected coverings of  $X$  classified by conjugacy classes of open subgroups of  $\pi_1^{top}(X, x)$ . Hence by Theorem 2.11 the result holds.  $\square$

**Theorem 2.20.** ([5]). *A covering space  $\tilde{X}$  of  $X$  is the Spanier space if and only if  $p_*\pi_1(\tilde{X}, \tilde{x}) = \pi_1^{sp}(X, x)$ , where  $p : \tilde{X} \rightarrow X$  is its corresponding covering map.*

Now, we are in a position to state and prove the main result of the paper.

**Theorem 2.21.** *All universal coverings of a connected and locally path connected space are Spanier spaces.*

*Proof.* If  $p : \tilde{X} \rightarrow X$  is a universal covering, then by the proof of Theorem 2.16, we have  $p_*\pi_1(\tilde{X}, \tilde{x}) = \pi_1^{sp}(X, x)$ . Hence Theorem 2.20 gives the result.  $\square$

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